I've got the (set of) power(s of two definable in Semënov arithmetic)!

Results from the master's project

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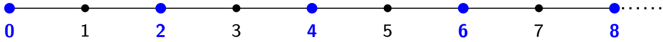
Today's talk: on definable sets

- Here we're going to be working in first-order logic, and we're going to be working with the natural numbers.
- First let's write down a definition.

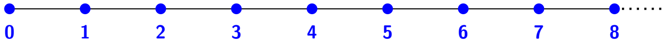
Definition

The set defined by a formula $\phi(x)$ is the set of numbers such that $\phi(x)$ is true.

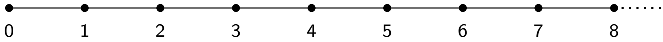
- So some examples:
  - The set defined by $\phi(x) := \exists y\ y + y = x$ is $\{0, 2, 4, 6, \ldots\}$.



  - The set defined by $\psi(x) := x = x$ is $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$.



  - The set defined by $\chi(x) := \forall y\ y \neq x$ is $\emptyset$.



- We can go on. Pick your favourite formula over the naturals and it defines some set.

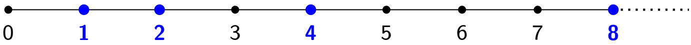
Two fragments of Semënov arithmetic

- Here we're interested in the theory of Presburger arithmetic; that's the language of first-order logic over natural numbers with addition.
- We have two extensions that we'd like to consider here.

Definition

The *prespower* theory is the extension of Presburger arithmetic with a predicate $2^{\mathbb{N}}(\cdot)$ for powers of two.

So for example we can define the set of powers of two:



Definition

The *lifted Presburger* theory is the extension of Presburger arithmetic with, for every formula $\phi(x)$ of Presburger arithmetic, a predicate $\text{Lift}_\phi(\cdot)$.

- What does the predicate $\text{Lift}_\phi(\cdot)$ do? Let's find out on the next slide.

The lifting predicate

- Given a formula ϕ, the $\mathrm{Lift}_\phi(x)$ predicate holds just when $\exists z\ x = 2^z \wedge \phi(z)$ holds.
- That means that $\mathrm{Lift}_\phi(x)$ holds just when x is a power of two and ϕ holds of the exponent of x.

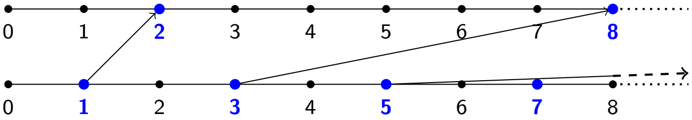


Figure 1: The lifting predicate applied to the set $\{1, 3, 5, 7, \ldots\}$.

- We can see that the lifted Presburger fragment can express anything the prespower fragment can:

Definition

If S is a set defined in the prespower fragment, then S is definable in the lifted Presburger fragment.

Proof.

We can emulate the powers of two predicate $2^{\mathbb{N}}(\cdot)$ using the formula $\mathrm{Lift}_{x=x}(x)$; that holds exactly when x is a power of two. □

- So anything that we can say in the prespower fragment we can say in the lifted Presburger fragment.
- Suprisingly, though, this holds in the other direction.

The magic of number theory

- To see why, we need to use some number theory. Let's try and define the lifted predicate $\text{Lift}_{\exists y\ y+y=x}(x)$.
- That defines the powers of four $\{2^2, 2^4, 2^6, 2^8, \ldots\}$.
- If we write that out another way, we get the following set:

$$\{4, 16, 64, 256, 1024, 4096, \ldots\}.$$

- If we write out all the powers of two that *aren't* powers of four, we get this other set:

$$\{1, 2, 8, 32, 128, 512, 2048, \ldots\}.$$

- You'll notice that everything in the first set ends in a 4 or a 6, and that nothing else in the second set does.

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	
1	2	4	8	6	2	4	8	6	

- This is the trick.

And it generalises

- We can therefore define the predicate $\text{Lift}_{\exists y\ y+y=x}(x)$ using just the predicate $2^{\mathbb{N}}(\cdot)$. Can we do this for **any** formula ϕ?
- Yes, using the power of number theory.
- So it follows that the prespower and the lifted Presburger fragments are equivalent.

Goodbye!

So, the main takeaways:

- Two logics that look like they say different things sometimes say the same thing.
- Number theory is useful even in computer science.
- Writing things out in full helps to understand them.

Thanks for listening!

Any questions?